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A GLOBALLY CONVERGENT BALL NEWTON METHOD. (U)

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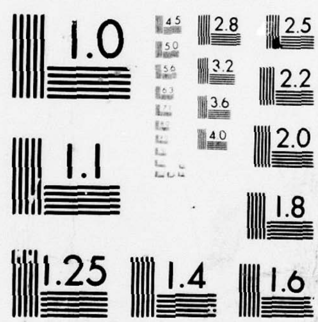

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(14) MRC-TSR-2022

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A GLOBALLY CONVERGENT BALL NEWTON METHOD

Karl L. Nickel*

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ABSTRACT

A new n-dimensional Newton method is presented. In each step a whole n-dimensional ball is determined rather than a single new approximation point. This ball contains the desired zero of the given function. The method is globally convergent. If the given initial ball does not contain any zero, then the method stops after a finite number of steps. Depending upon the assumptions which are made, the convergence of the ball radii is linear, superlinear or quadratic.

AMS(MOS) Subject Classifications: 30A08, 65H10

Key Words: Newton method, global convergence, error inclusion, interval analysis

Work Unit Number 2 - Other Mathematical Methods

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*This paper was stimulated by the "Symposium on Analysis and Computation of Fixed Points" at the University of Wisconsin-Madison, Madison, Wisconsin, U.S.A. in May of 1979. It was written while the author was visiting the Mathematics Research Center at the University of Wisconsin. Address of the author: Institut für Angewandte Mathematik der Universität, Hermann-Herder-Str. 10, D 7800 Freiburg i.Br., West Germany.

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Because (inclusion) balls are used, it is convenient to call the method a <u>ball</u> Newton method. Accordingly the "usual" Newton method in the following will be called a <u>point</u> Newton method.

The basic idea of the following method is the use of sets (balls) instead of points. This idea originates in interval mathematics. The first interval Newton method was given by R. E. Moore in [4]. It is easy to show that the Moore method is globally convergent for $n = 1$. This was shown in [5] for the first time. In the meantime, many interval variants of the Newton method have been investigated. A survey for $n = 1$ has been given by W. J. Beiser [2]. It seems, however, that in the case $n > 1$ no globally convergent interval Newton method has been found. There is indeed a criterion by Alefeld-Herzberger [1] which ensures global convergence for Moore's version. Nevertheless this criterion is rather unwieldy and is not satisfied in most practical cases.

For complex valued functions $f : B \subseteq \mathbb{C} \to \mathbb{C}$, complex discs can be used instead of complex intervals for the definition of a disc Newton method. This was done by P. Henrici [3] but no global convergence could be proven yet. It will be shown that the complex problem can be treated as a special case of the following general theory.

In order to define the point Newton method in general the following two assumptions are made:

$$\phi' \text{ exists on } B \;, \tag{6}$$

$$(\phi')^{-1} \text{ exists on } B \;. \tag{7}$$

In contrast to (6), <u>no</u> differentiability of f on B is demanded for the new method. Instead of (6), it suffices to assume the validity of a certain inclusion Lipschitz condition. This condition can be formulated especially simply for the case $n = 2$ and for $B \subseteq \mathbb{C}$. It then reads

$$f(x) - f(y) \in S(x - y) \quad \text{for } x, y \in B \;. \tag{8}$$

Herein $S := \{z \in \mathbb{C} \mid |s - z| \leqq \sigma|s|\}$ is a complex disc with midpoint s and radius $\sigma|s|$. It is assumed that $0 \leqq \sigma < 1$, i.e.

$$0 \notin S \quad . \tag{9}$$

These conditions (6), (7) for the point method and (8), (9) for the ball method can not be compared. On the one hand, the assumption (8) is weaker than (6), because f in (8) does not even have to be differentiable. On the other hand, (8) and (9) are stronger than (6), (7) because the data S, σ have to be known explicitly and because (6), (8) and (9) already imply (7).

The conditions to be imposed on f in $\mathbb{R}^n$ for $n > 2$ are still quite weak. They are not nearly as restrictive as the Kantorovich condition (see Ortega-Rheinboldt [7], p. 421). Even if f is analytic that condition is true in general only in a very small neighbourhood of a zero.

In what follows the following problem will not be considered: given a function $f : G \to \mathbb{R}^n$, find all zeros of f in G. If there is more than one zero in G and if G is convex then there are in general points in G where f' is not invertible. This can be shown already in the case $n = 1$ by the mean value theorem. Hence at these points the point Newton method (1), (2) cannot be applied. Therefore the Newton method always has to be combined with some other method if all the zeros of f on B are to be found. Actually, the new proposed ball Newton method also could be used for the solution of this more general problem. It not only guarantees the inclusion $\hat{z} \in Z_\nu$ for $\nu = 0, 1, \cdots$ but it also gives information regarding exclusions of the kind $\hat{z} \notin M$, where M is an appropriate set. With interval Newton methods these ideas have already been used successfully for the determination of all zeros of f in G, especially in the case $n = 1$. There will be another report upon the adoption of the ball Newton method to this more general problem.

It is therefore not the essential purpose of the following paper to compute all the zeros of a given function f. Rather, it will introduce a new family of Newton methods and will investigate their properties.

A GLOBALLY CONVERGENT BALL NEWTON METHOD

Karl L. Nickel*

1. Introduction.

In what follows the n-dimensional ball $B := \{z \in \mathbb{R}^n \,|\, \|b - z\| \leq \beta\} \subseteq \mathbb{R}^n$ is used as the basic domain. The zeros $\hat{z} \in B$ of the given function $f : B \to \mathbb{R}^n$ are wanted.

Let f be differentiable on B with the Jacobian ϕ'. Let the inverse $(\phi')^{-1}$ exist on B. Then the usual Newton method can be defined: The sequence $\{x_\nu\}$ of approximations $x_\nu \in \mathbb{R}^n$ is recursively constructed by

$$x_0 \in B \quad , \tag{1}$$

$$x_{\nu+1} := x_\nu - (\phi')^{-1}(x_\nu)\, f(x_\nu) \quad \text{for} \quad \nu = 0, 1, \cdots . \tag{2}$$

There are many variations of (1), (2). In the simplified Newton method (2) is replaced by

$$x_{\nu+1} := x_\nu - (\phi')^{-1}(x_0)\, f(x_\nu) \quad \text{for} \quad \nu = 0, 1, \cdots . \tag{3}$$

The literature dealing with the Newton method is huge. An introduction is given by L. Rall [8], and a survey may be found in the book of Ortega-Rheinboldt [7].

The methods (1), (2) and (1), (3) have the advantage that they are convergent in a neighborhood $U(\hat{z})$ of a zero $\hat{z}$ of f. This means that

$$x_0 \in U(\hat{z}) \tag{4}$$

implies $x_\nu \in U(\hat{z})$ for all $\nu = 0, 1, \cdots$ and that $\lim_{\nu\to\infty} x_\nu = \hat{z}$. This property is called local convergence. If ϕ' is continuous or if ϕ' satisfies a Lipschitz condition, then the order of convergence of (4), (2) is superlinear or quadratic.

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The disadvantages of (1), (2) and (1), (3) are well-known:

1) From (1) it may follow that

$$x_\nu \notin B \tag{5}$$

for some $\nu \in \mathbb{N}$. In this case the method cannot be applied further.

2) Even if $x_\nu \in B$ is true for all $\nu = 0, 1, \cdots$ the sequence $\{x_\nu\}$ is not necessarily convergent.

3) Without additional assumptions on f there are no easily determined a priori or a posteriori error bounds.

In order to avoid the first two disadvantages, modified Newton methods are sometimes considered instead of (1), (2). For example the new rule

$$x_{\nu+1} := x_\nu - \alpha_\nu (\phi')^{-1} (x_\nu)\, f(x_\nu) \quad \text{for} \quad \nu = 0, 1, \cdots$$

is often considered (see Ortega-Rheinbold [7]). Herein $\{\alpha_\nu\}$ is a sequence of appropriate real numbers often defined by minimization methods. If the α_ν are chosen suitably then (5) can be avoided and convergence for all $x_0 \in B$ can be enforced. This is called global convergence. Theoretically these results are very satisfying. Practically, their use is often quite awkward. Because of this and in order to remove the above third disadvantage, in what follows a new approach is presented.

The main idea of the new method is to construct a whole ball $Z_{\nu+1}$ with the property $\hat{z} \in Z_{\nu+1}$, instead of determining a new point $x_{\nu+1}$ in each step as in (2) or (3). Hence in each step a new a priori error bound is found. The initial ball is $Z_0 := B$. If $z_\nu \in Z_\nu \cap B$ is chosen, then case (5) can never occur. It will be shown that the radii of the balls Z_ν converge to zero at least like a geometrical sequence. Since $\hat{z} \in Z_\nu$ and $z_\nu \in Z_\nu$, $z_\nu \to \hat{z}$, hence global convergence occurs. Under appropriate assumptions on f, even superlinear or quadratic convergence can be shown. If there is no zero $\hat{z}$ of f in B then the method stops after a finite number of steps. This is an additional advantage of the new method.

SIGNIFICANCE AND EXPLANATION

One of the main problems in Numerical Analysis is the computation of the zeros of a n-dimensional function $f : \mathbb{R}^n \to \mathbb{R}^n$. For centuries, the so called Newton method has been used and investigated. This method has some very important advantages.

i) If it converges then its rate of convergence is usually quadratic.

ii) It is quite easy to describe, to understand and to program.

There are, however, very severe restrictions, drawbacks and disadvantages to this method:

i) In its pure form it is not a method at all nor does it provide an algorithm since in general one cannot predict a priori if it will work.

ii) It provides "approximations" but no error bounds, i.e. these "approximations" may have no significance at all.

iii) Even if it works and converges (which contrary to popular belief does not occur too often) it cannot give any information whether a zero lies in a given domain or if that domain is free of zeros.

The method described in this paper avoids all the above disadvantages without losing the advantages of the usual Newton method. It is based on the fundamental idea of interval mathematics: Instead of computing "approximation points" to the solution of a mathematical problem without known error bounds, one should try to evaluate "approximation sets" which are guaranteed to contain the solution. In the method used here, one of the most simple such sets is used: namely balls. All the disadvantages of the "point" Newton method disappear as soon as one switches to a "ball" Newton method. Astonishingly the assumptions to be made for the function f are not more complicated. In a certain sense they are even "simpler" and more "natural." Instead of differentiability of f a certain inclusion Lipschitz condition is required.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

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2. Notations and definitions; balls and regular ball operators.

2.1. Let $n \in \mathbb{N}$ be the fixed dimension number. Let an arbitrary norm $\|\cdot\|$ be chosen on $\mathbb{R}^n$. As matrix norm for $n \times n$-matrices the operator norm corresponding to $\|\cdot\|$ will be used. Only real values will be considered for simplicity. However, in examples the space $\mathbb{C}$ will be used also, as $\mathbb{C} \triangleq \mathbb{R}^2$.

Notations:

1) Lower case Greek letters $(\alpha, \beta, \cdots, \zeta, \zeta_\nu, \cdots)$ always mean real numbers.

2) Lower case Roman letters $(a, b, \cdots, z, z_\nu, \cdots)$ are n-dimensional vectors or vector valued functions. (Exception: n always means the integer dimension number.)

3) Capital Greek letters $(\Phi, \Sigma, \Lambda, \cdots)$ denote $n \times n$-matrices or matrix functions. As usual Λx, $\Sigma\Lambda$ means matrix multiplication.

4) Capital Roman letters $(A, B, \cdots, Z, Z_\nu, \cdots)$ are n-dimensional sets, in particular balls or ball valued functions. (Exception: I denotes the identity matrix.)

5) The function family considered is denoted by F.

Remark: In contrast to functional analysis, which uses different alphabets to distinguish between elements and operators, here different alphabets will identify points and operators which have values in different sets.

2.2. Definition (ball): Let $z \in \mathbb{R}^n$, $0 \leqq \zeta \in \mathbb{R}$. The set

$$Z := \{x \in \mathbb{R}^n \,|\, \|z - x\| \leqq \zeta\} \tag{10}$$

is called a (n-dimensional real) ball with the midpoint (center) z and the radius ζ.

Notations to (10):

$$\begin{aligned} Z &= \langle z, \zeta \rangle , \\ z &= \text{mid } Z , \\ \zeta &= \text{rad } Z , \\ x \in Z &:\Longleftrightarrow \|x - z\| \leqq \zeta . \end{aligned} \tag{11}$$

The midpoint of a ball which is characterized by a capital Roman letter will be called by the same - but lower case - Roman letter. For the radius, however, the corresponding small Greek letter is used. If $G \subseteq \mathbb{R}^n$ then the set of all such balls $Z \subseteq G$ will be denoted by $\mathbb{K}(G)$. Clearly $\mathbb{R}^n \subseteq \mathbb{K}(\mathbb{R}^n)$.

Properties:

1) Any ball $Z \in \mathbb{K}(\mathbb{R}^n)$ is compact and convex.

2) The inclusion $0 \in Z = \langle z, \zeta \rangle$ is true if and only if $\zeta \geqq \|z\|$.

Definition ($\mathbb{K}_{no}(\mathbb{R}^n)$): The set of all balls $Z \in \mathbb{K}(\mathbb{R}^n)$ for which $0 \notin Z$ holds is denoted by $\mathbb{K}_{no}(\mathbb{R}^n)$.

Properties:

1) Let be $Z \in \mathbb{K}(\mathbb{R}^n)$ and (differing from the notation in (11)) use the characterization

$$Z = \langle z, \zeta \|z\| \rangle$$

then the following holds: $Z \in \mathbb{K}_{no}(\mathbb{R}^n)$ iff $z \neq 0$ and $0 \leqq \zeta < 1$.

2) If $L = \langle \ell, \lambda \|\ell\| \rangle \in \mathbb{K}_{no}(\mathbb{R}^n)$, then the set of all balls $\langle \mu\ell, \lambda \|\mu\ell\| \rangle$, where $\mu \in \mathbb{R}$, lies in a (double) cone with the vertex at zero and with the defining angle $\alpha := \text{arc sin } \lambda$. See Figure 1 for $n = 2$.

Definition (ball arithmetic): Let $\alpha \in \mathbb{R}$, $a \in \mathbb{R}^n$, $Z = \langle z, \zeta \rangle \in \mathbb{K}(\mathbb{R}^n)$. Define

$$\alpha \cdot Z = Z \cdot \alpha := \langle \alpha z, |\alpha| \zeta \rangle , \tag{12}$$

$$a + Z = Z + a := \langle a + z, \zeta \rangle . \tag{13}$$

Further operations (e.g. $X + Y, \cdots$) are not needed in what follows and their algebraic structures are therefore not investigated here.

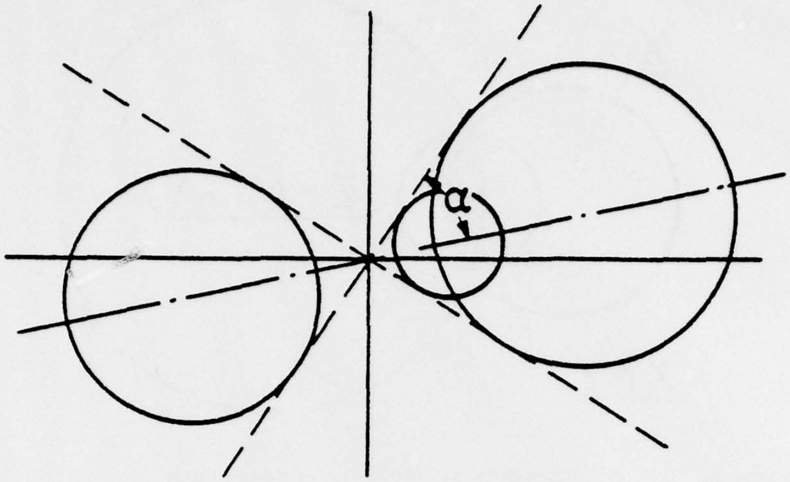


Figure 1. $n = 2$, Euclidean metric. The set of balls $\langle \mu\ell, \lambda\|\mu\ell\| \rangle$ for $\mu \in \mathbb{R}$ is contained in a (double) cone with the defining angle $\alpha := \arcsin \lambda$.

Properties:

1) If $\zeta = 0$ then (12) and (13) are the usual definitions on $\mathbb{R}^n$.

2) The following inclusions are true:

$$\alpha \cdot x \in \alpha \cdot Z \quad \text{for all} \quad x \in Z \ ,$$

$$a + x \in a + Z \quad \text{for all} \quad x \in Z \ .$$

Definition ($\langle X \cap Y \rangle$): Let $X, Y \in \mathbb{K}(\mathbb{R}^n)$ and let $X \cap Y \neq \emptyset$. Then there is (at least) one ball $Z \in \mathbb{K}(\mathbb{R}^n)$ for which $X \cap Y \subseteq Z$ and with $\operatorname{rad} Z = \min$. If there is more than one, an arbitrary one is selected. It is denoted by

$$Z = \langle X \cap Y \rangle \ .$$

See Figure 2 for $n = 2$ and for the Euclidean metric.

Properties: Let $X = \langle x, \xi \rangle$, $Y = \langle y, \eta \rangle$ and $Z = \langle X \cap Y \rangle = \langle z, \zeta \rangle$. Then the following inclusions and inequalities are true:

$$\left.\begin{aligned} & z \in X \ , \quad z \in Y \ , \\ & \zeta \leqq \min(\xi, \eta) \ . \end{aligned}\right\} \tag{14}$$

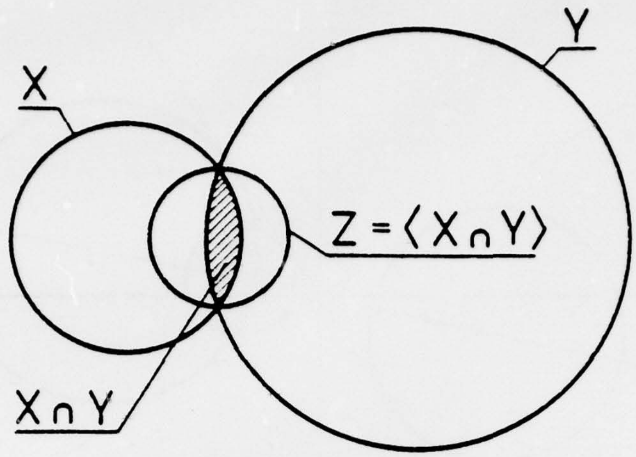


Figure 2. $n = 2$, Euclidean metric. The intersection $X \cap Y$ of the two balls X and Y is shaded. $Z = \langle X \cap Y \rangle$ is the smallest ball containing $X \cap Y$.

2.3. Definition (regular ball operator): Let Λ be a regular $n \times n$-matrix, and $\lambda \in \mathbb{R}$ with $0 \leqq \lambda < 1$. The regular ball operator L is defined by the ball valued operator

$$L : \mathbb{R}^n \to \mathbb{K}_{no}(\mathbb{R}^n) \cup \{0\} \quad ,$$

$$Lx := \langle \Lambda x, \lambda \|\Lambda x\| \rangle \quad \text{for} \quad x \in \mathbb{R}^n$$

Notations: The operator L will also be denoted by

$$L = \langle\langle \Lambda, \lambda \rangle\rangle \quad .$$

Remarks:

1) In what follows, more general ball valued operators $S : \mathbb{R}^n \to \mathbb{K}(\mathbb{R}^n)$ are not needed.

2) Any ball operator can also be considered as an operator ball according to

$$\left.\begin{aligned} L &= \langle\langle \Lambda, \lambda \rangle\rangle \\ &:= \{\psi \ n \times n\text{-matrix} \ \big| \ \|(\psi - \Lambda)x\| \leqq \lambda \|\Lambda x\| \quad \text{for all} \quad x \in \mathbb{R}^n\} \end{aligned}\right\} \tag{15}$$

3) This view and formula (15) are equivalent to

$$\psi x \in Lx := \langle \Lambda x, \lambda \|\Lambda x\| \rangle \quad \text{for all} \quad x \in \mathbb{R}^n \quad . \tag{16}$$

4) The inclusion (16) will be abbreviated to

$$\psi \in L \quad .$$

Properties: For any regular ball operator L the following is true:

1) $0 \notin Lx$ for $x \neq 0$.

2) $\psi \in L = \langle\langle \Lambda, \lambda \rangle\rangle$ if and only if

$$\| \psi \Lambda^{-1} - I \| = \| (\psi - \Lambda) \Lambda^{-1} \| \leqq \lambda \quad .$$

This means that the relative error between ψ and Λ is bounded by $\lambda < 1$.

3) If $\psi \in L = \langle\langle \Lambda, \lambda \rangle\rangle$ then ψ is a regular matrix and for the inverse ψ^{-1} the following inequality holds with cond $\Lambda := \|\Lambda\| \, \|\Lambda^{-1}\|$:

$$\| \psi^{-1} \Lambda - I \| = \| (\psi^{-1} - \Lambda^{-1}) \Lambda \| \leqq \frac{\lambda}{1 - \lambda} \text{ cond } \Lambda \quad . \tag{17}$$

This means again that the relative error between ψ^{-1} and Λ^{-1} is bounded. If in addition

$$\lambda < \frac{1}{1 + \text{cond } \Lambda}$$

holds then the right hand side of (17) is smaller than one, i.e.

$$\psi^{-1} \in L_1 := \langle\langle \Lambda^{-1}, \frac{\lambda}{1 - \lambda} \text{ cond } \Lambda \rangle\rangle \tag{18}$$

and L_1 is a regular ball operator.

3. <u>The class of functions</u> F.

<u>3.1 Definition</u> (class F): Let the class F of functions be the set of all functions $f : B \to \mathbb{R}^n$ which satisfy the following property: For each set C of the form

$$C = \langle a, \alpha \rangle \cap B \tag{19}$$

with $a \in B$, there exist a regular $n \times n$-matrix $\Lambda = \Lambda(C)$ and a real number $\lambda = \lambda(C)$ such that

$$0 \leqq \lambda < 1 \tag{20}$$

and for all $x, y \in C$,

$$\|x - y - \Lambda(f(x) - f(y))\| \leqq \lambda \|\Lambda(f(x) - f(y))\| \ . \tag{21}$$

If there is more than one $\lambda(C)$ and/or $\Lambda(C)$, an arbitrary one is selected and attached to F.

<u>Remarks</u>:

1) By using the regular ball operator

$$L := \langle\langle \Lambda, \lambda \rangle\rangle \tag{22}$$

the inequalities (20), (21) can also be written as

$$x - y \in L(f(x) - f(y)) \quad \text{for} \quad x, y \in C \ . \tag{23}$$

2) If $f \in F$ then the inverse f^{-1} of f exists on the set $f(B) := \{f(x) | x \in B\}$. Hence by defining $u := f(x)$, $v := f(y)$ the inclusion (23) can also be written as

$$f^{-1}(u) - f^{-1}(v) \in L(u - v) \ . \tag{24}$$

From (24) the class F of functions can also be characterized by the

<u>Equivalent Definition</u> (F): The class F consists of the set of all functions $f : B \to \mathbb{R}^n$ for which f^{-1} exists on $f(B)$ and which satisfy the inclusion Lipschitz condition (24) on C defined by (19).

<u>3.2.</u> With these results and with those of §2, the functions $f \in F$ have the following <u>Properties</u>:

1) There is at most one zero $\hat{z}$ of $f \in F$ in B.

2) On C defined by (19), any function $f \in F$ satisfies the Lipschitz condition

$$\frac{\|x - y\|}{(1 + \lambda)\|\Lambda\|} \leqq f(x) - f(y) \leqq \frac{\|\Lambda^{-1}\|}{1 - \lambda} \|x - y\| \quad .$$

Hence, f is continuous on B. There are similar Lipschitz conditions for f^{-1}.

3) Let $f \in F$ be Fréchet differentiable at the point $x \in B$ and let $\phi'(x)$ be the Jacobian. Then $(\phi')^{-1}(x)$ exists and is bounded by

$$\frac{1 - \lambda}{\|\Lambda^{-1}\|} \leqq \|(\phi')^{-1}(x)\| \leqq (1 + \lambda)\|\Lambda\| \quad .$$

Moreover, the relative error between Λ and $(\phi')^{-1}$ can be bounded by

$$\|((\phi')^{-1}(x) - \Lambda)\Lambda^{-1}\| = \|(\phi')^{-1}(x)\Lambda^{-1} - I\| \leqq \lambda \quad .$$

4) Let $f \in F$ be Fréchet differentiable on the entire domain C defined by (19). Then for all $x, y \in C$

$$\|\phi'(x) - \phi'(y)\| \leqq \frac{2\lambda}{1 - \lambda} \|\Lambda^{-1}\| \quad .$$

5) The result of 4) implies: Assume that for all sets $C := A \cap B$ where $a =$ mid $A \in B$ the following is true:

$$\left.\begin{aligned} &\|\Lambda^{-1}(C)\| \leqq \mu \quad , \\ &\lambda(C) \leqq \lambda_0 < 1 \quad , \end{aligned}\right\} \tag{25}$$

for appropriate real values $\mu, \lambda_0 \in \mathbb{R}$. If

$$\lambda(C) \to 0 \quad \text{for rad } A \to 0 \quad , \tag{26}$$

then ϕ' is continuous on B and if there exists $\sigma \in \mathbb{R}$ such that

$$\lambda(C) \leqq \sigma \cdot \text{rad } A \quad , \tag{27}$$

then ϕ' is Lipschitz continuous on B.

3.3. Criteria for $f \in F$.

Assume that there exists a matrix $\psi = \psi(x, y)$ $(= \psi(y, x))$ for all $x, y \in C$ defined by (19) such that

$$f(x) - f(y) = \psi(x - y) \quad . \tag{28}$$

This is certainly true if f satisfies one of the mean value theorems on C (see Ortega-Rheinboldt [7]). In general there are infinitely many such matrices. It will be shown in §4 how to construct such matrices ψ which are continuous.

3.3.1. Let there exist $\psi^{-1} = \psi^{-1}(x, y)$ on C and let the inclusion

$$\psi^{-1}(x, y) \in L \tag{29}$$

be satisfied for all $x, y \in C$ with the regular ball operator $L = \langle\langle \Lambda, \lambda \rangle\rangle$. Then (21) is true, i.e. $f \in F$.

3.3.2. Let there exist an $n \times n$-matrix Λ and a real constant $0 \leqq \lambda < 1$ such that for all $x, y \in C$

$$\|I - \Lambda\psi\| \leqq \lambda/(1 + \lambda) \quad . \tag{30}$$

Then Λ^{-1} and $\psi^{-1}(x, y)$ do exist on C, the operator $L := \langle\langle \Lambda, \lambda \rangle\rangle$ is regular and $\psi^{-1} \in L$. Hence $f \in F$ by 3.3.1.

3.3.3. Let f be continuously Fréchet differentiable on C with the Jacobian $\phi' = \phi'(x, y)$. For $x, y \in C$ define:

$$\psi(x, y) := \int_0^1 \phi'(x + t(y - x))dt \quad . \tag{31}$$

From the Mean Value Theorem and because $C := A \cap B$ is convex, the following two additional sufficient conditions arise:

3.3.3.1. Let there be a regular $n \times n$-matrix Λ and a real number $0 \leqq \sigma < 1/(1 + \text{cond } \Lambda)$ such that for all $x \in C$:

$$\phi'(x) \in L_2 := \langle\langle \Lambda^{-1}, \sigma \rangle\rangle \quad . \tag{32}$$

Then also $\psi(x, y) \in L_2$ for all $x, y \in C$, furthermore $\psi^{-1}(x, y)$ exists on C, and with the definition $\lambda := \frac{\sigma}{1 - \sigma}$ cond Λ, the inclusion $\psi^{-1} \in L := \langle\langle \Lambda, \lambda \rangle\rangle$ holds by (18). Then, $f \in F$ by (29).

3.3.3.2. Let there be an $n \times n$-matrix Λ and a real number $0 \leqq \lambda < 1$ such that for all $x \in C$:

$$\| I - \Lambda\phi'(x) \| \leqq \lambda/(1 + \lambda) \quad .$$

Then (30) holds, so Λ^{-1} and $\psi^{-1}(x, y)$ exist for $x, y \in C$ and $f \in F$ holds.

3.4. Examples: The case $n = 1$ and the complex case.

The two spaces $\mathbb{R}$ and $\mathbb{C}$ can be treated together. As usual $\mathbb{C}$ is imbedded in $\mathbb{R}^2$; furthermore on both $\mathbb{R}$ and $\mathbb{C}$ the Euclidean norm is used. In what follows, the case $n = 1$ is shown; the notations for $\mathbb{C}$ are then added in square brackets: [ ].

The ball $Z = \langle z, \zeta \rangle$ is an interval [a disc]. If $0 \notin Z$ then Z^{-1} exists and is an interval [a disc] which can be written as

$$Z^{-1} = \frac{|z|^2}{|z|^2 - \zeta^2} \langle z^{-1}, \zeta |z|^{-2} \rangle \quad .$$

If $x \in Z$ then also $x^{-1} \in Z^{-1}$.

The application of a regular ball operator $S = \langle\langle \Sigma, \sigma \rangle\rangle$ to an element x can be interpreted as real [complex] multiplication of the real interval $\langle 1, \sigma \rangle =$ $= [1 - \sigma, 1 + \sigma]$ by $0 \neq \Sigma \in \mathbb{R} [\in \mathbb{C}]$ and $x \in \mathbb{R} [\in \mathbb{C}]$ according to

$$Sx = \langle \Sigma x, \sigma |\Sigma x| \rangle = \Sigma \cdot \langle 1, \sigma \rangle \cdot x \quad . \tag{33}$$

For functions $f : B \to \mathbb{R}[\mathbb{C}]$ and C defined by (19) one has the

Criterion: On $\mathbb{R}[\mathbb{C}]$ the property $f \in F$ is true if and only if f satisfies on any set C the inclusion Lipschitz condition

$$f(x) - f(y) \in S(x - y) \qquad \text{for } x, y \in C \quad . \tag{8}$$

Herein $S = \Sigma \cdot \langle 1, \sigma \rangle$ is defined according to (33) with $0 \neq \Sigma \in \mathbb{R} [\in \mathbb{C}]$ and $0 \leqq \sigma < 1$. The ball operator $L = \langle\langle \Lambda, \lambda \rangle\rangle$ in (21) and (23) is defined by

$$\Lambda := \Sigma^{-1}/(1 - \sigma^2) \quad , \quad \lambda := \sigma \quad . \tag{34}$$

If f is differentiable on C [hence holomorphic] then (8) is satisfied for

$$\phi'(z) \in \Sigma \cdot \langle 1, \sigma \rangle \qquad \text{for } z \in C \quad . \tag{35}$$

The proofs are elementary [on $\mathbb{C}$ for example by using the inversion of complex circles] and will be omitted.

4. The ball Newton operator N.

Definition (ball Newton operator): Let $f \in F$, $A = \langle a, \alpha \rangle \in \mathbb{K}(\mathbb{R}^n)$ with $a \in B$. Let $L = \langle\langle \Lambda, \lambda \rangle\rangle$ be the regular ball operator corresponding to $C := A \cap B$ according to (22). Define the ball Newton operator N by

$$\left.\begin{aligned} &N : C \to \mathbb{K}(\mathbb{R}^n) \ , \\ &Nx := x - Lf(x) \\ &\quad = \langle x - \Lambda f(x) \ , \ \lambda\|\Lambda f(x)\| \rangle \quad \text{for } x \in C \ . \end{aligned}\right\} \tag{36}$$

Obviously $N = N(C)$, this dependence will be observed and used later occasionally.

Properties:

1) From (13) and by (11) it follows that

$$\text{mid } Nx = x - \Lambda f(x)$$

and (37)

$$\text{rad } Nx = \lambda\|\Lambda f(x)\| \ .$$

Hence both mid Nx and rad Nx are continuous on B.

2) If $\hat{z} \in C$ then:

$$f(\hat{z}) = 0 \text{ if and only if } \hat{z} = N(\hat{z}) \ .$$

Hence the zeros of f are on C exactly the fixed points of the operator N.

3) Test N (Non-existence): If $x \in C$ and

$$C \cap Nx = \emptyset$$

then there exists no zero of f on C.

4) Test E (Existence): Let $f \in F$ and assume that f is Fréchet differentiable on C. Let there be one point $x \in C$ such that

$$Nx \subseteq C \ . \tag{38}$$

Then, f has (exactly) one zero $\hat{z} \in C$.

5) If $x, \hat{z} \in C$ and $f(\hat{z}) = 0$ then

$$\hat{z} \in Nx \ . \tag{39}$$

This means: By applying the Newton operator N to an arbitrary point $x \in C$ one gets an error bound for the zero $\hat{z}$ of f in C. In other words: By applying the

Newton operator N no zero $\hat{z}$ of f can "get lost." However, $x \nsubseteq Nx$ if $x \neq \hat{z}$.

6) Let there exist a sequence $\{x_\nu\}$ with $x_\nu \in C$ for $\nu \in \mathbb{N}$ and with $\lim_{\nu\to\infty} x_\nu = \hat{z}$, where $f(\hat{z}) = 0$. Then the following is true:

$$\hat{z} \in Nx_\nu \quad \text{for } \nu \in \mathbb{N} ,$$

$$\lim_{\nu\to\infty} \operatorname{mid} Nx_\nu = \hat{z} \quad \text{and} \tag{40}$$

$$\lim_{\nu\to\infty} \operatorname{rad} Nx_\nu = 0 . \tag{41}$$

This means: The application of the Newton operator N to a convergent sequence $\{x_\nu\}$ gives a sequence of error bounds $\{Nx_\nu\}$ which converges also in the sense of (40) and (41).

The <u>proofs</u> of the properties 1) to 3) and 5), 6) will be omitted. Only the proof of <u>Test E</u> will be given. The basic idea of this proof has already been used in an earlier paper [6]. Define $x = (\xi_1, \cdots, \xi_n)$, $y = (\eta_1, \cdots, \eta_n)$, $f = (\varphi_1, \cdots, \varphi_n)$, $\varphi_{\mu\nu} := \partial\varphi_\mu/\partial\xi_\nu$. Let the matrix $\psi = \psi(x, y)$ have the components $\psi_{\mu\nu} = \psi_{\mu\nu}(\xi_1, \xi_2, \cdots, \xi_\nu, \eta_\nu, \eta_{\nu+1}, \cdots \eta_n)$ for $\mu, \nu = 1(1)n$. Let the $\psi_{\mu\nu}$ be defined by the following divided differences

$$\psi_{\mu\nu} := \begin{cases} \dfrac{\varphi_\mu(\xi_1,\cdots,\xi_{\nu-1},\xi_\nu,\eta_{\nu+1},\cdots,\eta_n)-\varphi_\mu(\xi_1,\cdots,\xi_{\nu-1},\eta_\nu,\eta_{\nu+1},\cdots,\eta_n)}{\xi_\nu - \eta_\nu} & \text{for } \xi_\nu \neq \eta_\nu \\[2ex] \varphi_{\mu\nu}(\xi_1,\cdots,\xi_{\nu-1},\xi_\nu,\eta_{\nu+1},\cdots,\eta_n) & \text{for } \xi_\nu = \eta_\nu . \end{cases}$$

The identity (28) is satisfied by the construction of ψ. If one of the arguments x or y is fixed, then $\psi(x, y)$ is continuous with respect to the other argument y or x on C. Because of $f \in F$, the inverse $\psi^{-1}(x, y)$ exists for all $x, y \in C$. Furthermore, $\psi^{-1}(x, y)$ is also partially continuous with respect to x or y on C. Let $x \in C$ be fixed. Define

$$g(y) := y - \psi^{-1}(x, y)f(y) \quad . \tag{42}$$

The function g is continuous for $y \in C$. Furthermore, by (28) the identity

$$g(y) = x - \psi^{-1}(x, y)f(x)$$

holds. By using (36), (29) and (38) one then gets

$$g(y) \in x - Lf(x) = Nx \subseteq C \quad .$$

Hence the continuous function g maps the convex and compact set C into itself. By the Schauder fixed point theorem, there is therefore at least one fixed point $\hat{y} = g(\hat{y}) \in C$. But because of (42) this fixed point $\hat{y}$ of g is also a zero of the function f. ∎

5. The simplified ball-Newton algorithm SNA.

5.1. Suppose $f \in F$.

Problem:

i) Is there a zero $\hat{z}$ of f on B?

ii) If yes, compute $\hat{z}$ in a constructive way.

How to proceed: A simplified ball Newton algorithm SNA will be presented. Starting with $Z_0 := B$ successive balls $Z_1, Z_2, \cdots$ will be constructed with the midpoints $z_\nu := \text{mid } Z_\nu \in B$ for $\nu = 0, 1, \cdots$.

Either the algorithm stops after a finite number of steps. This is true if and only if f does not have a zero $\hat{z} \in B$.

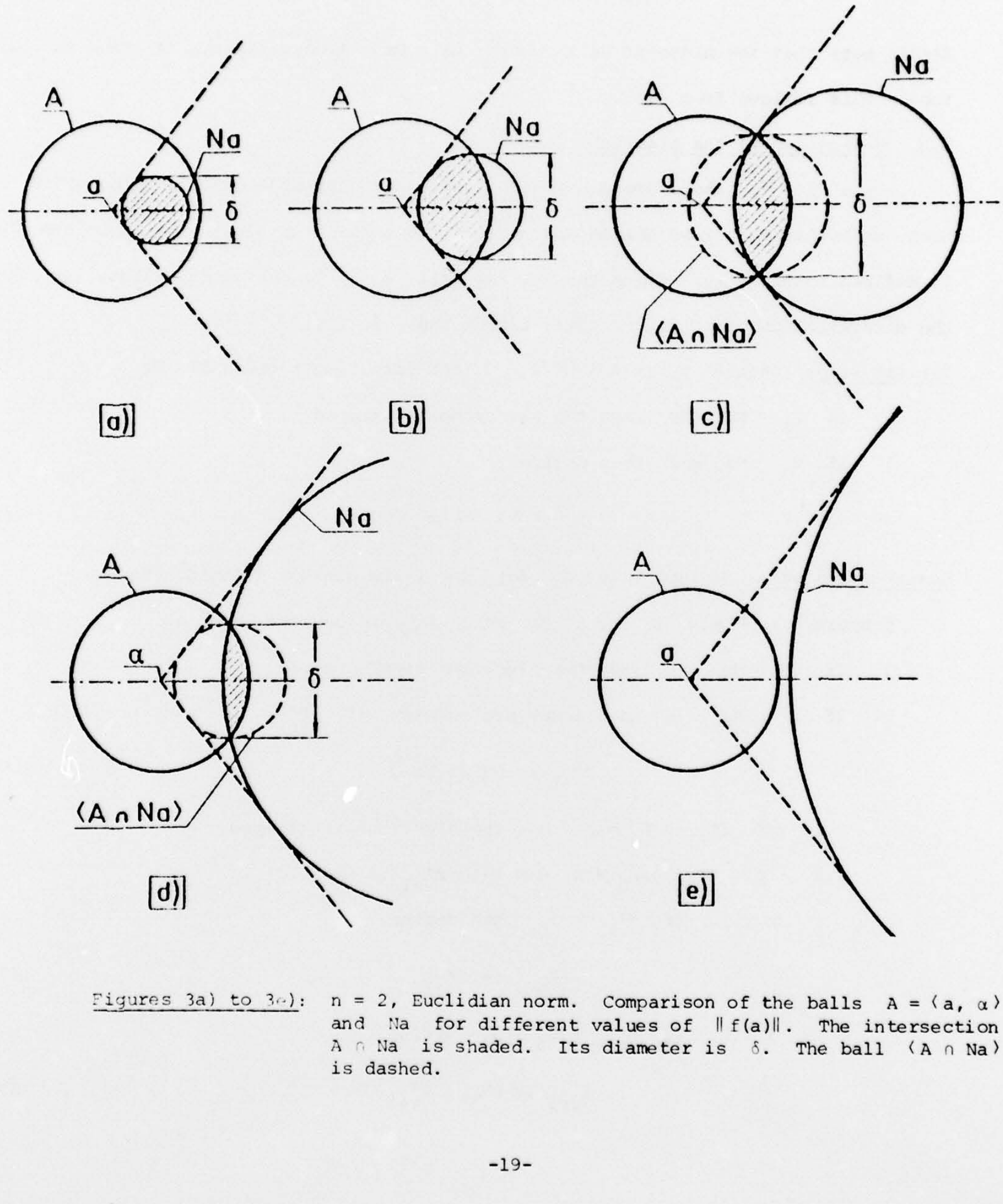
Or the construction can be continuated indefinitely. This happens if and only if there is a (uniquely determined) zero $\hat{z}$ of f on B. Then the midpoints z_ν converge to that zero $\hat{z}$. Furthermore, at each step the error inclusion $\hat{z} \in Z_\nu$ holds and the radii ζ_ν converge to zero.

During this simplified algorithm SNA the fixed initial parameters $\Lambda(B)$, $\lambda(B)$ are always used, notwithstanding the possibility that later on better data $\Lambda(Z_\nu)$, $\lambda(Z_\nu)$ could be available. Hence SNA corresponds to the simplified point Newton method (1), (3). Like the latter it is only linearly convergent. A more general Newton algorithm NA with updating in each step will later be presented in §6. It corresponds to the general point Newton method (1), (2) and, therefore, is superlinearly or even quadratically convergent.

5.2. Preliminary remark. Let $A = \langle a, \alpha \rangle$ be a ball and relative to the midpoint a define the new ball $Na := a - Lf(a)$ with the Newton operator N. Then, depending on the value of $\|f(a)\|$, different possibilities arise as sketched in Figure 3 for $n = 2$ and the Euclidean norm. As long as $\|f(a)\|$ is very small, the ball Na is completely contained in A. For growing values of $\|f(a)\|$ the radius of Na expands. The intersection $A \cap Na$, which is shaded in Figure 3, is first growing but shrinks later and finally disappears completely. Let δ be the diameter of the set $A \cap Na$.

Then by inspecting Figure 3, one sees that always

$$\delta \leqq 2\lambda\alpha \quad .$$



Figures 3a) to 3e): n = 2, Euclidian norm. Comparison of the balls $A = \langle a, \alpha \rangle$ and Na for different values of $\| f(a) \|$. The intersection $A \cap Na$ is shaded. Its diameter is δ. The ball $\langle A \cap Na \rangle$ is dashed.

The maximum is reached in figure 3c). By constructing the smallest ball $\langle A \cap Na \rangle$ which contains the set $A \cap Na$ (this ball is dashed in Figures 3c) to 3e)), one therefore gets for $\alpha \neq 0$

$$\delta/2 = \text{rad } \langle A \cap Na \rangle \leqq \lambda\alpha < \alpha \ . \tag{43}$$

Kindly note that the midpoint of the ball $\langle A \cap Na \rangle$ always lies in A (and in Na too). This follows from (14).

5.3. Definition of the algorithm SNA.

Since $f \in F$, there are constants $\Lambda(B)$, $\lambda(B)$ satisfying (20), (21). With these constants the fixed Newton operator N is defined by (36). The algorithm SNA is defined recursively. Since the initial ball $Z_0 := B$ has an exceptional position, the determination of Z_1 is different from that of $Z_2, Z_3, \dots$.

Initial step: Define $Z_0 := B = \langle z_0, \zeta_0 \rangle$ and furthermore the ball Nz_0.

1. If $Z_0 \cap Nz_0 = \emptyset$ then the algorithm is stopped.
2. If $Z_0 \cap Nz_0 \neq \emptyset$ then define

$$Z_1 := \langle Z_0 \cap Nz_0 \rangle \ . \tag{44}$$

Continuation step: Assume that Z_ν for $\nu \geqq 1$ is already defined with the property $z_\nu = \text{mid } Z_\nu \in Z_0$. The ball Nz_ν is derived from z_ν.

1. If $Z_\nu \cap Nz_\nu = \emptyset$ then the algorithm is stopped.
2. If $Z_\nu \cap Nz_\nu \neq \emptyset$ then a new preliminary ball $Z^*_{\nu+1}$ is constructed by

$$Z^*_{\nu+1} := \langle Z_\nu \cap Nz_\nu \rangle \ . \tag{45}$$

   2.1. If $Z^*_{\nu+1} \cap Z_0 = \emptyset$ then the algorithm is stopped.

   2.2. Let $Z^*_{\nu+1} \cap Z_0 \neq \emptyset$ and let $z^*_{\nu+1} := \text{mid } Z^*_{\nu+1}$.

   2.2.1. If $z^*_{\nu+1} \in Z_0$ then define

$$Z_{\nu+1} := Z^*_{\nu+1} \ . \tag{46}$$

   2.2.2. If $z^*_{\nu+1} \notin Z_0$ then define

$$Z_{\nu+1} := \langle Z_0 \cap Z^*_{\nu+1} \rangle \ . \tag{47}$$

5.4. Theorem 1: Suppose $f \in F$. The function f has no zero in B if and only if the algorithm SNA stops. The function f has a (uniquely determined) zero $\hat{z} \in B$ if and only if the algorithm SNA can be continued indefinitely. In this case it yields a sequence $\{Z_\nu\}$ of balls $Z_\nu = \langle z_\nu, \zeta_\nu \rangle$ for which

$$\lim_{\nu \to \infty} z_\nu = \hat{z} \in B$$

and (48)

$$\hat{z} \in Z_\nu \qquad \text{for } \nu = 0, 1, \cdots .$$

The sequence of the ball radii converges at least linearly to zero according to

$$\lim_{\nu \to \infty} \zeta_\nu = 0 \quad ,$$

$$\zeta_{\nu+1} \leqq \lambda \zeta_\nu \qquad \text{for } \nu = 0, 1, \cdots . \tag{49}$$

Proof:

1) If $\lambda = 0$ then f is a linear function by (21). In this case the application of the ball Newton operator N to z_0 yields the point $\hat{z}$. If $\hat{z} \notin B$ then the algorithm stops at the initial step. If $\hat{z} \in B$, then the algorithm never stops and gives the result $Z_\nu = \langle \hat{z}, 0 \rangle$ for all $\nu = 1, 2, \cdots$. In what follows, therefore, $\lambda > 0$ can be assumed with loss of generality.

2) Assume that there is a (uniquely determined) zero $\hat{z}$ of f on $Z_0 := B$. Then $\hat{z} \in Nz_0$ by (39). Therefore, $\hat{z} \in Z_1$ also, where Z_1 is defined by (44). Hence, the algorithm is not stopped during the initial step.

Assume that it is already proved that $\hat{z} \in Z_\nu$. By construction, $z_\nu \in Z_0$. Therefore, $\hat{z} \in Nz_\nu$ by (39) and this implies $\hat{z} \in Z^*_{\nu+1}$, where $Z^*_{\nu+1}$ is defined by (45). Hence, neither $Z_\nu \cap Nz_\nu = \emptyset$ nor $Z^*_{\nu+1} \cap Z_0 = \emptyset$ and therefore the algorithm is not stopped at the ν-th step either.

This is true for all $\nu = 0, 1, \cdots$ and means: If $\hat{z} \in B$, then the ball Newton algorithm SNA does not stop.

3) Conversely, assume now that the algorithm SNA does not stop. It is to be shown that the sequence $\{z_\nu\}$ of the ball midpoints converges to a limit point $\hat{z} \in B$.

Initial step: From the definition (44), one deduces because of $z_1 \in Z_0$ and because of (43) the two inequalities

$$\|z_1 - z_0\| \leqq \zeta_0 \quad , \tag{50}$$

$$\zeta_1 \leqq \lambda \zeta_0 \quad . \tag{51}$$

Continuation step: By inserting (43) in the definition (45) and observing (14) one gets

$$\|z^*_{\nu+1} - z_\nu\| \leqq \zeta_\nu \quad , \tag{52}$$

$$\zeta^*_{\nu+1} \leqq \lambda \zeta_\nu \quad . \tag{53}$$

In the case 2.2.1. because of (46) this gives

$$\|z_{\nu+1} - z_\nu\| \leqq \zeta_\nu \quad , \tag{54}$$

$$\zeta_{\nu+1} \leqq \lambda \zeta_\nu \quad . \tag{55}$$

In the case 2.2.2., one deduces from the definition (47) by observing (14) the inequalities

$$\|z_{\nu+1} - z^*_{\nu+1}\| \leqq \zeta^*_{\nu+1} \quad , \tag{56}$$

$$\zeta_{\nu+1} \leqq \zeta^*_{\nu+1} \quad . \tag{57}$$

The inequalities (52), (56) and (53) together with the triangle inequality therefore yield

$$\begin{aligned} \|z_{\nu+1} - z_\nu\| &\leqq \zeta_\nu + \zeta^*_{\nu+1} \\ &\leqq \zeta_\nu (1 + \lambda) \quad . \end{aligned} \tag{58}$$

The inequality (57) together with (53) lead to the previous inequality (55).

These proven inequalities (51) for $\nu = 0$ and (55) for $\nu \geqq 1$ are exactly the inequality (49) claimed in Theorem 1. By using induction one gets from this that

$$\zeta_\nu \leqq \zeta_0 \lambda^\nu \qquad \text{for } \nu = 0, 1, \cdots \quad . \tag{59}$$

By combining the inequalities (50), (54) and (58) and by observing (59), one gets for all $\nu = 0, 1, \cdots$ the bounds

$$\|z_{\nu+1} - z_\nu\| \leqq \zeta_0 (1 + \lambda)\lambda^\nu$$

and from this for all $\mu > \nu$,

$$\|z_\mu - z_\nu\| \leqq \zeta_0 \frac{1 + \lambda}{1 - \lambda} \lambda^\nu \quad .$$

Hence, $\{z_\nu\}$ is a Cauchy sequence because $\lambda < 1$. Therefore, z_ν converges to a limit point $\hat{z}$. This proves (48) because $z_\nu \in Z_0$ for $\nu = 0, 1, \cdots$.

4) It remains to be shown that this limit point $\hat{z}$ in (48) is a zero of f. In order to show this, one defines the set of balls $\{Nz_\nu\}$ corresponding to the midpoints z_ν of the balls Z_ν. It is assumed that the algorithm SNA does not stop, hence $Z_\nu \cap Nz_\nu \neq \emptyset$. By this and by the triangle inequality the radii of Nz_ν can be bounded by

$$\text{rad } Nz_\nu \leqq \zeta_\nu \lambda/(1 - \lambda) \quad . \tag{60}$$

On the other hand, one sees from (37) that

$$\text{rad } Nz_\nu = \lambda\|\Lambda f(z_\nu)\| \geqq \lambda\|\Lambda^{-1}\|^{-1} \|f(z_\nu)\| \quad . \tag{61}$$

By inserting (61) and (59) in (60) one finally gets for $\nu = 0, 1, \cdots$

$$\|f(z_\nu)\| \leqq \zeta_0 \frac{\|\Lambda^{-1}\|}{1 - \lambda} \lambda^{\nu+1} \quad .$$

Since $\lambda < 1$, this gives immediately $\lim_{\nu\to\infty} f(z_\nu) = 0$. Because of the continuity of f and because of (48) the limit point $\hat{z}$ is therefore a (namely, the) zero of f in B.

5) This exhausts all logical possibilities. Hence Theorem 1 is proved. ■

5.5. With this procedure and these results an easy answer can be given to the

Extended Problem: Let $f \in F$, assume $A = \langle a, \alpha\rangle \in \mathbf{K}(\mathbb{R}^n)$, $a \in B$, $C := \langle A \cap B\rangle$. Decide if there is a zero $\hat{z}$ of f on C. If yes, then compute $\hat{z}$ in a constructive way.

Solution: Replace the initial step in the algorithm SNA by $Z_0 := B$, $Z_1 := C$. Subsequently define $Z_2, Z_3, \cdots$ as above by the continuation step. Then Theorem 1 remains valid if one replaces B by C.

6. The general ball Newton algorithm NA.

Definition (NA): The initial step is carried out as in the algorithm SNA. It gives $Z_0 := B$ and Z_1 by (44). Assume that the ball Z_ν is already determined for $n \geqq 1$. Define $C_\nu := Z_0 \cap Z_\nu$ and choose $\lambda(C_\nu)$, $\Lambda(C_\nu)$ such that (20), (21) are satisfied. Define the operator N by (36) with these constants. Carry out the continuation step as in the algorithm SNA.

Remark: At each step the "full available information" is exploited for updating. Hence, the algorithm NA corresponds to the point Newton algorithm (1), (2).

Theorem 2: Let $f \in F$ and assume (25) and either

a) the condition (26) or

b) the condition (27).

Then, all the statements of Theorem 1 also remain true for the algorithm NA. In addition to (49) the following holds: The convergence of the ball radii ζ_ν is either

a) superlinear or

b) quadratic.

The proof follows immediately from the definitions (25) to (27).

7. Examples.

The following two examples have been chosen deliberately to be simple. The space $\mathbb{C} \doteq \mathbb{R}^2$ is used. In both cases, the function $f : \mathbb{C} \to \mathbb{C}$ is a quadratic polynomial, and its zeros are therefore known. Furthermore the derivative ϕ' of f is linear. If $C \in \mathbb{K}(\mathbb{C})$ is a disc then the set $\phi'(C) := \{\phi'(z) \mid z \in C\}$ is also a disc. Hence the condition (35) can be verified extremely easily.

7.1. Non-existence example.

Let $f(z) := (z - 3 + 2i)(z - 3 - 2i) = z^2 - 6z + 13$ and $B := \langle 0, 2 \rangle$. Clearly $\phi'(B) = \langle -6, 4 \rangle$. By (34), (35) one gets therefore $f \in F$ with $\Lambda(B) = -3/10$, $\lambda(B) = 2/3$.

By using the simplified ball Newton algorithm SNA one gets (see Figure 4): $Z_0 = \langle 0, 2 \rangle$, $Nz_0 = \langle 3.9, 2.6 \rangle$, $Z_1 = \langle 83/52, \sqrt{3927}/52 \rangle$, $Nz_1 = \langle 18319/5408, 3229/2704 \rangle$ $\subseteq \langle 3.4, 1.3 \rangle$. Hence $Z_0 \cap Nz_1 = \emptyset$ and the algorithm SNA is stopped. By the non existence test N and Theorem 1 there is therefore no zero $\hat{z}$ of f in $Z_0 = B$.

7.2. Existence example.

Let $f(z) := (z - 1)(z - 4) = z^2 - 5z + 4$ and $B := \langle 0, 2 \rangle$. Clearly $\phi'(B) = \langle -5, 4 \rangle$. By (34), (35) one therefore gets $f \in F$ with $\Lambda(B) = -5/9$ and $\lambda(B) = 4/5$. The discs $Z_0 := B, Z_1, \cdots, Z_4, Nz_0, \cdots, Nz_3$ of the algorithm SNA are sketched in Figures 5 and 6. One sees immediately that $Nz_1 \subseteq Z_0$. Hence by the existence test E there exists (exactly) one zero $\hat{z} \in Nz_1 \subseteq Z_0 = B$. The speed of convergence of the simplified algorithm SNA is very low. See the values in table 1 where the discs Z_0 to Z_4 are given. After 4 iterations not even the first digit of $\hat{z}$ is ensured.

The circles $\phi'(Z_\nu)$ are sketched in Figure 7 for $\nu = 0, 1, 2$. One sees that $0 \in \phi'(Z_1)$ and that $\phi'(Z_0) \cap \phi'(Z_1) = \phi'(Z_0)$. Hence, the improved algorithm NA cannot give better values than the algorithm SNA before $\nu = 2$. Starting with the index $\nu = 2$, the results are actually much better due to the inclusion $\phi'(Z_2) \subset \phi'(Z_0)$. This can be seen from the values in table 1 for the discs Z_0 to Z_4.

From the values of the radii $\zeta_2, \zeta_3, \zeta_4$ one can already anticipate quadratic convergence. After 4 iterations the quadratically convergent algorithm NA already guarantees nearly 5 digits of $\hat{z}$. After one more iteration (which is not included in table 1) even 10 digits can be guaranteed.

Table 1. Comparison of the discs Z_0 to Z_4 created by the algorithms SNA and NA for the example 7.2.

	Simplified ball Newton algorithm SNA, linearly convergent		Ball Newton algorithm NA, quadratically convergent	
Z_0	$\langle 0$	$, 2 \rangle$	$\langle 0$	$, 2 \rangle$
Z_1	$\langle 1.3$	$, 1.519 \cdots \rangle$	$\langle 1.3$	$, 1.519 \cdots \rangle$
Z_2	$\langle 0.85$	$, 0.36 \rangle$	$\langle 0.85$	$, 0.36 \rangle$
Z_3	$\langle 1.1125$	$, 0.21 \rangle$	$\langle 1.000338 \cdots$	$, 0.032801 \cdots \rangle$
Z_4	$\langle 0.9320 \cdots$	$, 0.1443 \cdots \rangle$	$\langle 0.99999979 \cdots$	$, 0.00000740 \cdots \rangle$

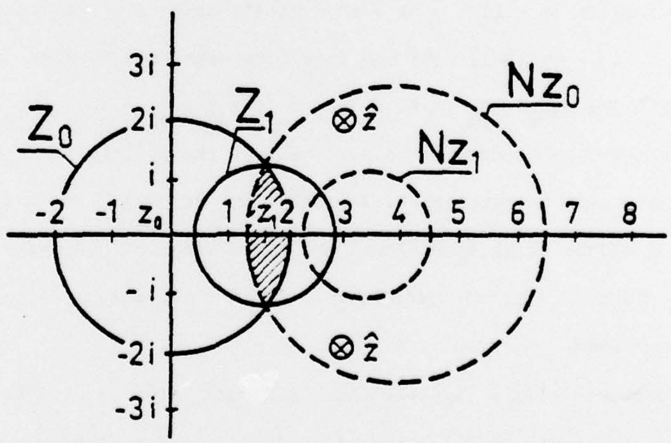


Figure 4. Example 7.1 The four discs Z_0, Nz_0, Z_1, Nz_1. The set $Z_0 \cap Nz_0$ is shaded. Since $Z_0 \cap Nz_1 = \emptyset$ there is no zero $\hat{z} \in Z_0$.

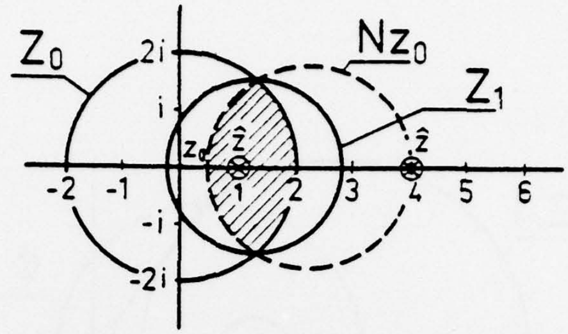


Figure 5. Example 7.2. Sketched are the three discs Z_0, Nz_0, Z_1. The set $Z_0 \cap Nz_0$ is shaded. The zero $\hat{z} = 1$ lies in Z_0, Nz_0, $Z_0 \cap Nz_0$ and in Z_1.

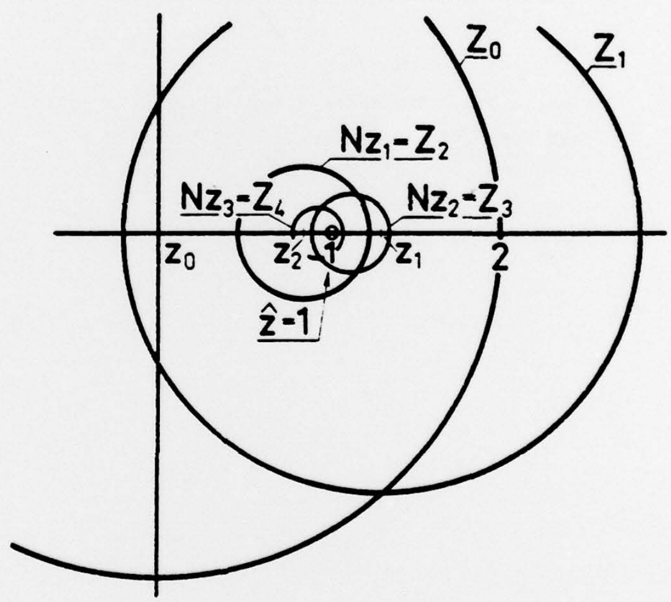
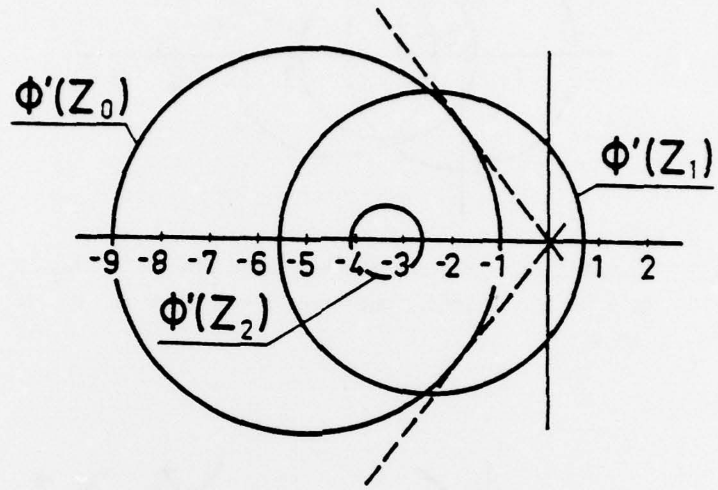


Figure 6. Example 7.2, larger scale than Figure 5. The discs Z_0 to Z_4 are shown. Note that $Z_2 = Nz_1 \subseteq Z_0$. Hence by test E there exists a zero $\hat{z}(= 1)$ with $\hat{z} \in Nz_1 \subseteq Z_0$.



<u>Figure 7</u>. Example 7.2. The three discs $\phi'(Z_0)$ to $\phi'(Z_2)$ are shown. Note that $0 \in \phi'(Z_1)$ and that $\phi'(Z_2) \subseteq \phi'(Z_0)$.

REFERENCES

[1] Alefeld, G. und J. Herzberger: Über das Newton-Verfahren bei nichtlinearen Gleichungssystemen. Z. Angew. Math. Mech 50 (1970), 773-774.

[2] Beiser, W. J.: Intervall-Newton-Verfahren zur Bestimmung von Nullstellen reeller Funktionen einer Veränderlichen. Freiburger Intervall-Berichte 79/2, (1979).

[3] Henrici, P.: Circular arithmetic and the determination of polynomial zeros. "Conference on Application of Numerical Analysis," Lecture Notes in Mathematics 228, Springer Verlag, 86-92 (1971).

[4] Moore, R. E.: Interval Analysis. Prentice-Hall, Inc., Englewood Cliffs, N.J. (1966).

[5] Nickel, K.: Triplex - ALGOL and applications. "Topics in Interval Analysis," ed. by E. Hansen, Oxford University Press, 10-24 (1969).

[6] Nickel, K.: On the Newton method in Interval Analysis. MRC Technical Summary Report #1136, University of Wisconsin, Madison, Wisconsin (1971).

[7] Ortega, J. M. and W. C. Rheinboldt: Iterative solution of nonlinear equations in several variables. Academic Press (1970).

[8] Rall, L.: Computational solution of nonlinear operator equations. Wiley, New York (1969).

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A new n-dimensional Newton method is presented. In each step a whole n-dimensional ball is determined rather than a single new approximation point. This ball contains the desired zero of the given function. The method is globally convergent. If the given initial ball does not contain any zero, then the method stops after a finite number of steps. Depending upon the assumptions which are made, the convergence of the ball radii is linear, superlinear or quadratic.

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